

A Geometrical Formulation of the Renormalization Group Method for Global Analysis

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Abstract

On the basis of the classical theory of envelope, we formulate the renormalization group (RG) method for global analysis, recently proposed by Goldenfeld et al. It is clarified why the RG equation improves things.

1 Introduction

Recently, Goldenfeld et al[1] have proposed a new method based on the renormalization group (RG) equation [2, 3] to get the asymptotic behavior of solutions of differential equations. The method is simple and has a wide variety of applications including singular and reductive perturbation problems in a unified way. However, the reason is obscure why the RG equation can be relevant and useful for global analysis: The RG equation is usually related with the scale invariance of the system under consideration. The equations which can be treated in the RG method are not confined to those with scale invariance[1]. Actually, what the RG method does in [1] may be said to construct an approximate but *global* solution from the ones with a local nature which were obtained in the perturbation theory; the RG equation is used to improve the global behavior of the local solutions. This fact suggests that the RG method can be formulated in a purely mathematical way without recourse to the concept of the RG. A purpose of the paper is to show that this is the case, thereby reveal the mathematical structure of the method.

Our formulation is based on the classical theory of envelopes [4]. As everybody knows, the envelope of a family of curves or surfaces has usually an improved global nature compared with the curves or surfaces in the family. So it is natural that the theory of envelopes may have some power for global analysis. One will recognize that the powerfulness of the RG equation in global analysis and also in the quantum field theory[2, 3] is due to the fact that

it is essentially an envelope equation. We shall also give a proof as to why the RG equation can give a globally improved solution to differential equations.

In the next section, a short review is given on the classical theory of envelopes, the notion of which is essential for the understanding of the present paper. In §3, we formulate the RG method in the context of the theory of envelopes and give a foundation to the method. In §4, we show a couple of other examples to apply our formulation. The last section is devoted to a brief summary and concluding remarks.

2 A short review of the classical theory of envelopes

To make the discussions in the following sections clear, we here give a brief review of the theory of envelopes. Although the theory can be formulated in higher dimensions[4], we take here only the one-dimensional envelopes. i.e., curves, for simplicity.

Let $\{C_\tau\}_\tau$ be a family of curves parametrized by τ in the x - y plane; here C_τ is represented by the equation

$$F(x, y, \tau) = 0. \quad (2.1)$$

We suppose that $\{C_\tau\}_\tau$ has the envelope E , which is represented by the equation

$$G(x, y) = 0. \quad (2.2)$$

The problem is to obtain $G(x, y)$ from $F(x, y, \tau)$.

Now let E and a curve C_{τ_0} have the common tangent line at $(x, y) = (x_0, y_0)$, i.e., (x_0, y_0) is the point of tangency. Then x_0 and y_0 are functions of τ_0 ; $x_0 = \phi(\tau_0)$, $y_0 = \psi(\tau_0)$, and of course $G(x_0, y_0) = 0$. Conversely, for each point (x_0, y_0) on E , there exists a parameter τ_0 . So we can reduce the problem to get τ_0 as a function of (x_0, y_0) ; then $G(x, y)$ is obtained as $F(x, y, \tau(x, y)) = G(x, y)$.¹ $\tau_0(x_0, y_0)$ can be obtained as follows.

The tangent line of E at (x_0, y_0) is given by

$$\psi'(\tau_0)(x - x_0) - \phi'(\tau_0)(y - y_0) = 0, \quad (2.3)$$

while the tangent line of C_{τ_0} at the same point reads

$$F_x(x_0, y_0, \tau_0)(x - x_0) + F_y(x_0, y_0, \tau_0)(y - y_0) = 0. \quad (2.4)$$

Here $F_x = \partial F / \partial x$ and $F_y = \partial F / \partial y$. Since both equations must give the same line,

$$F_x(x_0, y_0, \tau_0)\phi'(\tau_0) + F_y(x_0, y_0, \tau_0)\psi'(\tau_0) = 0. \quad (2.5)$$

¹Since there is a relation $G(x_0, y_0) = 0$ between x_0 and y_0 , τ_0 is actually a function of x_0 or y_0 .

On the other hand, differentiating $F(x(\tau_0), y(\tau_0), \tau_0) = 0$ with respect to τ_0 , one has

$$F_x(x_0, y_0, \tau_0)\phi'(\tau_0) + F_y(x_0, y_0, \tau_0)\psi'(\tau_0) + F_{\tau_0}(x_0, y_0, \tau_0) = 0, \quad (2.6)$$

hence

$$F_{\tau_0}(x_0, y_0, \tau_0) \equiv \frac{\partial F(x_0, y_0, \tau_0)}{\partial \tau_0} = 0. \quad (2.7)$$

One can thus eliminate the parameter τ_0 to get a relation between x_0 and y_0 ,

$$G(x, y) = F(x, y, \tau_0(x, y)) = 0, \quad (2.8)$$

with the replacement $(x_0, y_0) \rightarrow (x, y)$. $G(x, y)$ is called the discriminant of $F(x, y, t)$.

Comments are in order here: (i) When the family of curves is given by the function $y = f(x, \tau)$, the condition Eq.(2.7) is reduced to $\partial f / \partial \tau = 0$; the envelope is given by $y = f(x, \tau_0(x))$. (ii) The equation $G(x, y) = 0$ may give not only the envelope E but also a set of singularities of the curves $\{C_\tau\}_\tau$. This is because the condition that $\partial F / \partial x = \partial F / \partial y = 0$ is also compatible with Eq. (2.7).

As an example, let

$$y = f(x, \tau) = e^{-\epsilon\tau}(1 - \epsilon \cdot (x - \tau)) + e^{-x}. \quad (2.9)$$

Note that y is unbound for $x - \tau \rightarrow \infty$ due to the secular term.

The envelope E of the curves C_τ is obtained as follows: From $\partial f / \partial \tau = 0$, one has $\tau = x$. That is, the parameter in this case is the x -coordinate of the point of the tangency of E and C_τ . Thus the envelope is found to be

$$y = f(x, x) = e^{-\epsilon x} - e^{-x}. \quad (2.10)$$

One can see that the envelope is bound even for $x \rightarrow \infty$. In short, we have obtained a function as the envelope with a better global nature from functions which are bound only locally.

As an illustration, we show in Fig.1 some of the curves given by $y = f(x, \tau_0)$ together with the envelope.

Fig.1

3 Formulation of the RG method based on the theory of envelopes

In this section, we formulate and give a foundation of the RG method[1] in the context of the classical theory of envelopes sketched in the previous section. Our formulation also includes an improvement of the prescription.

Although the RG method can be applied to both (non-linear) ordinary and partial differential equations, let us take the following simplest example to show our formulation:

$$\frac{d^2x}{dt^2} + \epsilon \frac{dx}{dt} + x = 0, \quad (3.1)$$

where ϵ is supposed to be small. The solution to Eq.(3.1) reads

$$x(t) = A \exp(-\frac{\epsilon}{2}t) \sin(\sqrt{1 - \frac{\epsilon^2}{4}}t + \theta), \quad (3.2)$$

where A and θ are constant to be determined by an initial condition.

Now, let us blindly try to get the solution in the perturbation theory, expanding x as

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots, \quad (3.3)$$

where x_n ($n = 0, 1, 2, \dots$) satisfy

$$\ddot{x}_0 + x_0 = 0, \quad \ddot{x}_{n+1} + x_{n+1} = -\dot{x}_n. \quad (3.4)$$

Thus $x_0 = A_0 \sin(t + \theta_0)$, and then $\ddot{x}_1 + x_1 = -A_0 \cos(t + \theta_0)$, and so on. Then we get for x_1 and x_2 as special solutions

$$\begin{aligned} x_1(t) &= -\frac{A_0}{2} \cdot (t - t_0) \sin(t + \theta_0), \\ x_2(t) &= \frac{A_0}{8} \{(t - t_0)^2 \sin(t + \theta_0) - (t - t_0) \cos(t + \theta_0)\}. \end{aligned} \quad (3.5)$$

Here we have intentionally omitted the unperturbed solution from $x_n(t)$ ($n = 1, 2, \dots$). Although this prescription is not adopted in [1], the proceeding calculations are simplified with this prescription; see also §4.² It should be noted that the secular terms have appeared in the higher order terms, which are absent in the exact solution and invalidates the perturbation theory for t far away from t_0 .

Inserting Eq.(3.5) into Eq.(3.3), we have

$$\begin{aligned} x(t, t_0) &= A_0 \sin(t + \theta_0) - \epsilon \frac{A_0}{2} (t - t_0) \sin(t + \theta_0) \\ &\quad + \epsilon^2 \frac{A_0}{8} \{(t - t_0)^2 \sin(t + \theta_0) - (t - t_0) \cos(t + \theta_0)\} + O(\epsilon^3) \end{aligned} \quad (3.6)$$

Now we have a family of curves $\{C_{t_0}\}_{t_0}$ given by functions $\{x(t, t_0)\}_{t_0}$ parametrized with t_0 . They are all solutions of Eq. (3.1) up to $O(\epsilon^3)$, but only valid locally, i.e., for t near t_0 . Let us find a function $x_E(t)$ representing the envelope E of $\{C_{t_0}\}_{t_0}$.

²It is amusing to see that the unperturbed solution in the higher order terms x_n ($n = 1, 2, \dots$) is analogous to the “dangerous” term in the Bogoliubov’s sense in the quantum-field theory of superfluidity and superconductivity[5].

According to the previous section, we only have to eliminate t_0 from

$$\frac{\partial x(t, t_0)}{\partial t_0} = 0, \quad (3.7)$$

and insert the resultant $t_0(t)$ into $x(t, t_0)$. Then we identify as $x_E(t) = x(t, t_0(t))$. It will be shown that $x_E(t)$ satisfies the original differential equation Eq. (3.1) *uniformly* for $\forall t$ up to $O(\epsilon^4)$; see below.

Eq.(3.7) is in the same form as the RG equation, hence the name of the RG method[1]. In our formulation, this is a condition for constructing the envelope.

Here comes another crucial point of the method. We assume that A_0 and B_0 are functionally dependent on t_0 ;

$$A_0 = A_0(t_0), \quad \theta_0 = \theta_0(t_0), \quad (3.8)$$

accordingly $x(t, t_0) = x(t, A_0(t_0), \theta_0(t_0), t_0)$. Then it will be found that Eq. (3.7) gives a complicated equation involving $A_0(t_0), \theta_0(t_0)$ and their derivatives as well as t_0 . It turns out, however, that one can actually greatly reduce the complexity of the equation by assuming that the parameter t_0 coincides with the point of tangency, that is ,

$$t_0 = t, \quad (3.9)$$

because $A_0(t_0)$ and $\theta_0(t_0)$ can be determined so that $t_0 = t$. We remark here that the meaning of setting $t_0 = t$ is not clearly explained in [1], while in our case, the setting has the clear meaning to choose the point of tangency at $t = t_0$.³

From Eq.'s (3.7) and (3.9), we have

$$\frac{dA_0}{dt_0} + \epsilon A_0 = 0, \quad \frac{d\theta_0}{dt_0} + \frac{\epsilon^2}{8} = 0. \quad (3.10)$$

Solving the simple equations, we have

$$A_0(t_0) = \bar{A}e^{-\epsilon t_0/2}, \quad \theta_0(t_0) = -\frac{\epsilon^2}{8}t_0 + \bar{\theta}, \quad (3.11)$$

where \bar{A} and $\bar{\theta}$ are constant numbers. Thus we get

$$x_E(t) = x(t, t) = \bar{A} \exp(-\frac{\epsilon}{2}t) \sin((1 - \frac{\epsilon^2}{8})t + \bar{\theta}). \quad (3.12)$$

Noting that $\sqrt{1 - \epsilon^2/4} = 1 - \epsilon^2/8 + O(\epsilon^4)$, one finds that the resultant envelope function $x_E(t)$ is an approximate but *global* solution to Eq.(3.1); see Eq. (3.2). In short, the solution obtained in the perturbation theory with the local nature has been “improved” by the envelope equation to become a global solution.

³It is interesting that the procedure to get the envelope of $x(t, A_0(t_0), \theta_0(t_0), t_0)$ assuming a functional dependence of A_0 and θ_0 on t_0 is similar to the standard prescription in which the general solution of a partial differential equation of first order is constructed from the complete solution.[4]

There is another version of the RG method[1], which involves a “renormalization” of the parameters. We shift the parameter for the local curves as follows: Let τ be close to t , and write $t - t_0 = t - \tau + \tau - t_0$. Then putting that

$$\begin{aligned} A(\tau) &= A_0(t_0)Z(t_0, \tau), \quad Z(t_0, \tau) = 1 - \frac{\epsilon}{2}(\tau - t_0) + \frac{\epsilon^2}{8}(\tau - t_0)^2, \\ \theta(\tau) &= \theta_0(t_0) + \delta\theta, \quad \delta\theta = -\frac{\epsilon^2}{8}(\tau - t_0), \end{aligned} \quad (3.13)$$

we have

$$\begin{aligned} x(t, \tau) &= A(\tau) \sin(t + \theta(\tau)) - \epsilon \frac{A(\tau)}{2} (t - \tau) \sin(t + \theta(\tau)) \\ &\quad + \epsilon^2 \frac{A(\tau)}{8} \{(t - \tau)^2 \sin(t + \theta(\tau)) - (t - \tau) \cos(t + \theta(\tau))\} + O(\epsilon^3), \end{aligned} \quad (3.14)$$

where

$$x(\tau, \tau) = A(\tau) \sin(\tau + \theta(\tau)). \quad (3.15)$$

Then the envelope of the curves given by $\{x(t, \tau)\}_\tau$ will be found to be the same as given in Eq. (3.12).

This may concludes the account of our formulation of the RG method based on the classical theory of envelopes. However, there is a problem left: Does $x_e(t) \equiv x(t, t)$ indeed satisfy the original differential equation? In our simple example, the result Eq.(3.12) shows that it does. It is also the case for all the resultant solutions worked out here and in [1]. We are, however, not aware of a general proof available to show that the envelope function should satisfy the differential equation (uniformly) up to the same order as the local solutions do locally. We give here a proof for that for a wide class of linear and non-linear ordinal differential equations (ODE). The proof can be easily generalized to partial differential equations (PDE).

Let us assume that the differential equation under consideration can be converted to the following coupled equation of *first order*:

$$\frac{d\mathbf{q}(t)}{dt} = \mathbf{F}(\mathbf{q}(t), t; \epsilon), \quad (3.16)$$

where ${}^t\mathbf{q} = (q_1, q_2, \dots)$ and \mathbf{F} are column vectors. It should be noted that \mathbf{F} may be a non-linear function of \mathbf{q} and t , although in our example,

$$q_1 = x, \quad q_2 = \dot{x}, \quad \mathbf{F} = \begin{pmatrix} q_2 \\ -q_1 - \epsilon q_2 \end{pmatrix}, \quad (3.17)$$

i.e., \mathbf{F} is linear in \mathbf{q} . We also assume that we have an approximate local solution $\tilde{\mathbf{q}}(t, t_0)$ around $t = t_0$ up to $O(\epsilon^n)$;

$$\frac{d\tilde{\mathbf{q}}}{dt} = \mathbf{F}(\tilde{\mathbf{q}}, t; \epsilon) + O(\epsilon^n). \quad (3.18)$$

One can see for our example to satisfy this using Eq.(3.6).

The envelope equation implies

$$\frac{\partial \tilde{\mathbf{q}}(t, t_0)}{\partial t_0} = 0 \quad (3.19)$$

at $t_0 = t$. With this condition, $\mathbf{q}_E(t)$ corresponding to $x_E(t)$ is defined by

$$\mathbf{q}_E(t) = \tilde{\mathbf{q}}(t, t). \quad (3.20)$$

It is now easy to show that $\mathbf{q}_E(t)$ satisfies Eq.(3.16) up to the same order as $\tilde{\mathbf{q}}(t, t_0)$ does: In fact, for $\forall t_0$

$$\left. \frac{d\mathbf{q}_E(t)}{dt} \right|_{t=t_0} = \left. \frac{d\tilde{\mathbf{q}}(t, t_0)}{dt} \right|_{t=t_0} + \left. \frac{\partial \tilde{\mathbf{q}}(t, t_0)}{\partial t_0} \right|_{t=t_0} = \left. \frac{d\tilde{\mathbf{q}}(t, t_0)}{dt} \right|_{t=t_0}, \quad (3.21)$$

where Eq.(3.19) has been used. And noting that $\mathbf{F}(\mathbf{q}_E(t_0), t_0; \epsilon) = \mathbf{F}(\mathbf{q}(t_0, t_0), t_0; \epsilon)$, we see for $\forall t$

$$\frac{d\mathbf{q}_E}{dt} = \mathbf{F}(\mathbf{q}_E(t), t; \epsilon) + O(\epsilon^n), \quad (3.22)$$

on account of Eq. (3.18). This completes the proof. It should be stressed that Eq.(3.22) is valid uniformly for $\forall t$ in contrast to Eq.(3.18) which is valid only locally around $t = t_0$.

4 Examples

Let us take a couple of examples to apply our formulation. These can be converted to equations in the form given in Eq. (3.16).

4.1 A boundary-layer problem

The first example is a typical boundary-layer problem[6]:

$$\epsilon \frac{d^2 y}{dx^2} + (1 + \epsilon) \frac{dy}{dx} + y = 0, \quad (4.1)$$

with the boundary condition $y(0) = 0, \quad y(1) = 1$. The exact solution to this problem is readily found to be

$$y(x) = \frac{\exp(-x) - \exp(-x/\epsilon)}{\exp(-1) - \exp(-1/\epsilon)}. \quad (4.2)$$

Now let us solve the problem in the perturbation theory. Introducing the inner variable X by $\epsilon X = x$ [6], and putting $Y(X) = y(x)$, the equation is converted to the following;

$$\frac{d^2 Y}{dX^2} + \frac{dY}{dX} = -\epsilon \left(\frac{dY}{dX} + Y \right). \quad (4.3)$$

Expanding Y in the power series of ϵ as $Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \dots$, one has

$$\begin{aligned} Y_0'' + Y_0' &= 0, \\ Y_1'' + Y_1' &= -Y_0' - Y_0, \\ &\vdots \end{aligned} \quad (4.4)$$

Here, $Y' \equiv dY/dX$ etc. To solve the equation, we set a boundary condition to $Y(X)$ and $Y_0(X)$ at $X = X_0$;

$$Y(X) = Y_0(X_0) = A_0, \quad (4.5)$$

where X_0 is an arbitrary constant and A_0 is supposed to be a function of X_0 .

For this problem, we shall follow the prescription given in [1] for the higher order terms. Then the solutions to these equations may be written as

$$\begin{aligned} Y_0(X) &= A_0 - B_0 e^{-(X-X_0)}, \\ Y_1(X) &= -A_0(X - X_0) - (B_0 + C_0)(e^{-(X-X_0)} - 1). \end{aligned} \quad (4.6)$$

Defining $A = A_0 + \epsilon(B_0 + C_0)$ and $B = B_0 + \epsilon(B_0 + C_0)$, we have

$$Y(X, X_0) = A - B e^{-(X-X_0)} - \epsilon A(X - X_0) + O(\epsilon^2). \quad (4.7)$$

In terms of the original coordinate,

$$y(x, x_0) = Y(X, X_0) = A - B e^{-(x-x_0)/\epsilon} - A(x - x_0) + O(\epsilon^2), \quad (4.8)$$

with $x_0 = X_0/\epsilon$.

Now let us obtain the envelope $Y_E(X)$ of the family of functions $\{Y(X, X_0)\}_{X_0}$ each of which has the common tangent with $Y_E(X)$ at $X = X_0$. According to the standard procedure to obtain the envelope, we first solve the equation,

$$\frac{\partial Y}{\partial X_0} = 0, \quad \text{with } X_0 = X, \quad (4.9)$$

and then identify as $Y(X, X) = Y_E(X)$.

Eq. (4.9) claims

$$A' + \epsilon A = 0, \quad B' + \epsilon B = 0, \quad (4.10)$$

with the solutions $A(X) = \bar{A} \exp(-\epsilon X)$, $B(X) = \bar{B} \exp(-X)$, where \bar{A} and \bar{B} are constant. Thus one finds

$$Y_E(X) = Y(X, X) = A(X) - B(X) = \bar{A} e^{-\epsilon X} - \bar{B} e^{-X}. \quad (4.11)$$

In terms of the original variable x ,

$$y_E(x) \equiv Y_E(X) = \bar{A} \exp(-x) - \bar{B} \exp(-\frac{x}{\epsilon}). \quad (4.12)$$

It is remarkable that the resultant $y_E(x)$ can admit both the inner and outer boundary conditions simultaneously; $y(0) = 1$, $y(1) = 1$. In fact, with the boundary conditions we have $\bar{A} = \bar{B} = 1/(\exp(-1) - \exp(-1/\epsilon))$, hence $y_E(x)$ coincides with the exact solution $y(x)$ given in Eq. (4.2).

In Fig. 2, we show the exact solution $y(x)$ and the local solutions $y(x, x_0)$ for several x_0 : One can clearly see that the exact solution is the envelope of the curves given by $\{y(x, x_0)\}_{x_0}$.

Fig.2

A comment is in order here: If we adopted the prescription given in §3 for the higher order terms, the perturbed solution $Y_1(X)$ reads $Y_1(X) = -A_0(X - X_0)$; note the boundary condition Eq.(4.5). Then the proceeding calculations after Eq. (4.6) would be slightly simplified.

4.2 A non-linear oscillator

In this subsection, we consider the following Rayleigh equation[6, 1],

$$\ddot{y} + y = \epsilon(\dot{y} - \frac{1}{3}\dot{y}^3). \quad (4.13)$$

Applying the perturbation theory with the expansion $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots$, one has

$$\begin{aligned} y(t, t_0) = & R_0 \sin(t + \theta_0) + \epsilon \left\{ \left(\frac{R_0}{2} - \frac{R_0^3}{8} \right) (t - t_0) \sin(t + \theta_0) \right. \\ & \left. + \frac{R_0^3}{96} (\cos 3(t + \theta_0)) \right\} + O(\epsilon^2). \end{aligned} \quad (4.14)$$

Here we have not included the terms proportional to the unperturbed solution in the higher order terms in accordance with the prescription given in §3, so that the following calculation is somewhat simplified than in [1]. Furthermore, the result with this prescription will coincide with the one given in the Krylov-Bogoliubov-Mitropolsky method[7], as we will see in Eq. (4.18).

Eq. (4.14) gives a family of curves $\{C_{t_0}\}_{t_0}$ parametrized with t_0 . The envelope E of $\{C_{t_0}\}_{t_0}$ with the point of tangency at $t = t_0$ can be obtained as follows:

$$\frac{\partial y(t, t_0)}{\partial t_0} = 0, \quad (4.15)$$

with $t_0 = t$. Assuming that \dot{R}_0 and $\dot{\theta}_0$ are $\sim O(\epsilon)$ at most, we have

$$\dot{R}_0 = \epsilon\left(\frac{R_0}{2} - \frac{R_0^3}{8}\right), \quad \dot{\theta}_0 = 0, \quad (4.16)$$

the solution of which reads

$$R_0(t) = \frac{\bar{R}_0}{\sqrt{\exp(-\epsilon t) + \bar{R}_0^2(1 - \exp(-\epsilon t))/4}}, \quad (4.17)$$

with $\bar{R}_0 = R_0(0)$ and $\theta_0 = \text{constant}$. Thus the envelope is given by

$$y_E(t) = y(t, t) = R_0(t) \sin(t + \theta_0) + \epsilon \frac{R_0(t)^3}{96} (\cos 3(t + \theta_0)) + O(\epsilon^2). \quad (4.18)$$

This is an approximate but global solution to Eq. (4.13) with a limit cycle in accordance with the result given in [7]. We note that since Eq.(4.13) can be rewritten in the form of Eq.(3.16), Eq.(4.18) satisfies Eq.(4.13) up to $O(\epsilon^2)$.

5 A brief summary and concluding remarks

We have given a geometrical formulation of the RG method for global analysis recently proposed by Goldenfeld et al[1]: We have shown that the RG equation can be interpreted as an envelope equation, and given a purely mathematical foundation to the method. We have also given a proof that the envelope function satisfies the same differential equation up to the same order as the functions representing the local curves do.

It is important that a geometrical meaning of the RG equation even in a generic sense has been clarified in the present work. The RG equation appears in various fields in physics. For example, let us take a model in the quantum field theory[8];

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + \text{c.t.}, \quad (5.1)$$

where c.t. stands for counter terms. The true vacuum in the quantum field theory is determined by the minimum of so called the effective potential $\mathcal{V}(\phi_c)$ [8, 9]. In the one-loop approximation, the renormalized effective potential reads

$$\mathcal{V}(\phi_c, M) = \frac{\lambda}{4!}\phi_c^4 + \frac{\lambda^2 \phi_c^4}{256\pi^2} \left(\ln \frac{\phi_c^2}{M^2} - \frac{25}{6} \right), \quad (5.2)$$

where M^2 is the renormalization point. To see a correspondence to the envelope theory, one may parametrize as $\phi_c^2 = \exp t$ and $M^2 = \exp t_0$, then one sees that $\ln \phi_c^2/M^2$ becomes a secular term $t - t_0$. In the quantum field theory, one applies the RG equation to improve the effective potential as follows[8, 10];

$$\frac{\partial \mathcal{V}}{\partial M^2} = 0, \quad \text{with } M^2 = \phi_c^2. \quad (5.3)$$

One sees that this is the envelope equation! The resultant “improved” effective potential is found to be

$$\mathcal{V}_{\text{impr}}(\phi_c) = \mathcal{V}(\phi_c, \phi_c) = \frac{\frac{\lambda}{4!}\phi_c^4}{1 - \frac{3\lambda}{16\pi^2} \ln \frac{\phi_c}{\phi_{c0}}}. \quad (5.4)$$

Thus one can now understand that the “improved” effective potential is nothing but the envelope of the effective potential in the perturbation theory. One also sees the reason why the RG equation with $\phi_c = M$ can “improve” the effective potential. Then what is the physical significance of the envelope function $\mathcal{V}_{\text{impr}}(\phi_c)$? One can readily show that for $\forall M$

$$\left. \frac{\partial \mathcal{V}(\phi_c)_{\text{impr}}}{\partial \phi_c^2} \right|_{\phi_c=M} = \left. \frac{\partial \mathcal{V}(\phi_c, M)}{\partial \phi_c^2} \right|_{\phi_c=M}, \quad (5.5)$$

owing to the envelope condition Eq. (5.3). This implies, for example, that the vacuum condensate ϕ_c that is given by $\partial \mathcal{V}_{\text{impr}} / \partial \phi_c^2 = 0$ is correct up to the same order of \hbar -expansion in which the original effective potential is calculated; this is irrespective of how large is the resultant ϕ_c . Detailed discussions of the application of the envelope theory to the quantum field theory will be reported elsewhere[11].⁴

The RG equation has also a remarkable success in statistical physics especially in the critical phenomena [3]. One may also note that there is another successful theory of the critical phenomena called coherent anomaly method (CAM)[13]. The relation between CAM and the RG equation theory is not known. Interestingly enough, CAM utilizes *envelopes* of susceptibilities and other thermodynamical quantities as a function of temperature. It might be possible to give a definite relation between CAM and the RG theory because the RG equation can be interpreted as an envelope equation, as shown in this work.

Mathematically, it is most important to give a rigorous proof for the RG method in general situations and to clarify what types of differential equations can be analyzed in this method, although we have given a simple proof for a class of ODE’s. We note that the proof can be generalized to partial differential equations, especially of first order with respect to a variable[12]. One should be also able to estimate the accuracy of the envelope theory for a given equation. We hope that this paper may stimulate studies for a deeper understanding of global analysis based on the theory of envelopes.

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⁴Recent renewed interest in “improving” the effective potential is motivated by the problem of how to “improve” the effective potentials with multi scales as appear in the standard model[10]. The observation given here that the RG equation can be interpreted as the envelope equation may give an insight into how to construct effective potentials with a global nature in multi-scale cases.

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Figure Captions

Fig.1 A family of functions and its envelope:

The thin lines show $y = \exp(-\epsilon\tau_0)(1 - \epsilon(x - \tau_0)) + \exp(-x)$ with $\tau_0 = 0.2, 0.4, 0.4, 0.6$ and 0.8 , which are attached to the respective lines. The thick line shows the envelope $y = \exp(-\epsilon x) - \exp(-x)$. ($\epsilon = 0.8$.)

Fig. 2 The thin lines show $y(x, x_0) = A(x_0) - B(x_0) \exp(-(x - x_0)/\epsilon) - A(x_0)(x - x_0)$ with $x_0 = 0.2, 0.4$ and 0.8 , which are attached to the respective lines. The thick line shows $y(x)$ given in Eq.(4.2). ($\epsilon = 0.1$.)